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Technical Report No. 32-431 (Part III)

A Simplified Statistical Model for Missile Launching—III

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PASADENA, CALIFORNIA

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Carleton B. Solloway

T. W. Hamilton

T. W. Hamilton, Chief,
Systems Analysis Section

JET PROPULSION LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CALIFORNIA

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ABSTRACT

The two previous reports of this series consider the problem of estimating the number of days necessary to complete the countdown procedures for the launching of three vehicles from two launch pads when simultaneous countdowns are permitted but simultaneous launchings are disallowed. In the present Report, simultaneous countdowns are disallowed under any conditions as are simultaneous launchings. The model for the delays encountered does not differ from the previous two.

I. INTRODUCTION

This Report is the third of a series in which we analyze the problem of estimating the number of days it takes to launch three vehicles from two launch pads. In the previous two reports,* simultaneous countdowns were permitted to occur, although simultaneous launches were prohibited for obvious practical reasons. It is conceivable that the facilities available will be inadequate for conducting simultaneous countdowns (as is now the case). For this reason, we wish to investigate the same fundamental question when simultaneous countdowns are not permitted.

In the present Report, we solve this problem, still retaining the same model for the nature of the delays expected to be encountered as in the previous reports. This delay model is extremely simple. In future reports, the delay model will be generalized to conform to a more realistic (and necessarily much more complex) situation. Further reports are also anticipated in which the problem will be generalized in several directions, namely (1) increasing the number of vehicles, (2) increasing the number of launch pads, (3) generalizing the countdown procedures, and (4) generalizing the delay models as our knowledge of the process increases.

*Solloway, C. B., *A Simplified Statistical Model for Missile Launching—I*, Technical Report No. 32-431 (Part I), Jet Propulsion Laboratory, Pasadena, May 1, 1963.
Solloway, C. B., *A Simplified Statistical Model for Missile Launching—II*, Technical Report No. 32-431 (Part II), Jet Propulsion Laboratory, Pasadena, June 1, 1963.

II. THE PROBLEM

In this Report, we are concerned with the number of days necessary to complete the countdown procedures for the launching of three vehicles from two pads under the following assumptions:

1. Two vehicles are erected simultaneously on two pads, and the countdown proceeds on one vehicle.
2. When the countdown has been successfully completed on the first vehicle, the countdown on the second vehicle is initiated the following day.
3. Simultaneously, the vacated pad is immediately cleaned and prepared for the third vehicle. There is a (fixed) period of R days' delay after a launching before the same pad may be utilized for a second launch attempt (the turnaround time).
4. After the third vehicle has been erected on the vacated pad, the countdown procedure is *not* initiated until the day after the second vehicle is launched.
5. Each vehicle is independent of, and identical to, the others. On any single countdown attempt, there is a probability p of a successful completion and a probability $q = 1 - p$ of failure. Any failure results in the termination of that countdown attempt, and a new attempt is made *the following day;* that is, any failure leads to a one-day delay. It is assumed that a successful countdown attempt can be completed in one day.
6. The failure to complete a countdown does not affect the subsequent attempts in any way. That is, the trials are independent from day to day as well as from vehicle to vehicle.

Fundamentally, the model discussed here differs from those discussed in Parts I and II only in that a simultaneous countdown is not permitted at any time.

III. THE PRINCIPAL RESULTS — EXACT EXPRESSIONS

Let N be the number of days until the third successful countdown. Then, the exact frequency function for N is given by

$$f(N) = \text{probability of completing the third countdown on the } N\text{th day}$$

$$= p^2 p^{N-R-2} (1 - q^{R-1}) (N - R - 1) + p^3 q^{N-3} \; \frac{(N-R)(N-R-1)}{2} \qquad N \geq R + 2 \tag{1}$$

The cumulative distribution function for N is given by

$$F(N) = \text{probability of completing the third countdown on or before the } N\text{th day}$$

$$= \sum_{x=R+2}^{N} f(x)$$
$$= \left\{ \begin{array}{l} (1 - q^{R-1}) \left[1 - q^{N-R} - (N-R)\, pq^{N-R-1}\right] \\ + \dfrac{q^{R-1}}{2} \left[2 - 2q^{N-R+1} - 2(N-R+1)\, pq^{N-R} - (N-R+1)(N-R)\, p^2 p^{N-R-1}\right] \end{array} \right\} \tag{2}$$

and the moment generating function $M(\theta)$ for N is given by

$$M(\theta) = \sum_{x=R+2}^{\infty} e^{\theta x} f(x)$$
$$= e^{\theta(R+2)} \left[\frac{p^2(1-q^{R-1})}{(1-e^{\theta}q)^2} + \frac{p^3 q^{R-1}}{(1-e^{\theta}q)^3}\right] \tag{3}$$

from which we obtain the mean μ_N and variance σ_N^2 in the usual manner. Thus,

$$\mu_N = \left.\frac{dM(\theta)}{d\theta}\right|_{\theta=0} = (R+2) + \frac{2q}{p} + \frac{q^R}{p}$$
$$\sigma_N^2 = \left.\frac{dM(\theta)}{d\theta^2}\right|_{\theta=0} - \mu_N^2 \tag{4}$$

The variance is complicated; its approximation is given in Section IV.

Figures 1 and 2 show the exact cumulative distribution function $F(N)$ for $p = 0.2$, 0.3, and 0.4 for $R \geq 18$ and $= 1$, respectively. For large $R (\geq 18)$, the curves are indistinguishable as a function of R, so they are plotted as a function of $N-R$.

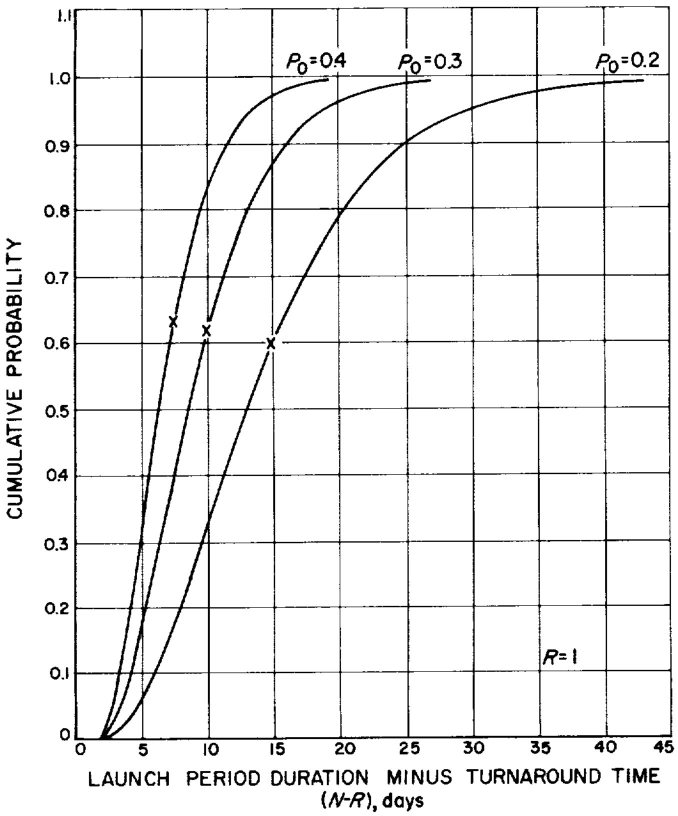

Fig. 1. Cumulative probability of launching three vehicles from two pads, $R \geq 18$

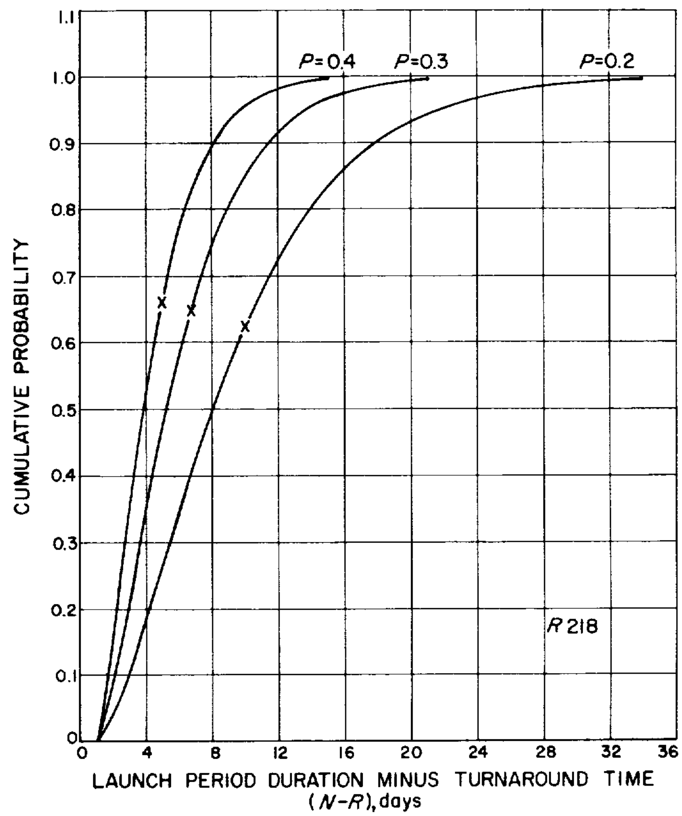

Fig. 2. Cumulative probability of launching three vehicles from two pads, $R = 1$

IV. APPROXIMATE EXPRESSIONS

The expressions in Eq. (1) to (4) are rather complicated. In most practical situations, R is quite large (e.g., an optimistically small realistic R today is about 18 days), and even with a small probability of success p, major simplifications can be obtained. The realism of the model does not justify the accuracy necessary to include these terms. Making the approximations, we obtain

$$f(N) \simeq p^2 q^{N-R-2}(N - R - 1) \tag{5}$$

$$F(N) \simeq 1 - q^{N-R} - (N - R)\, pq^{N-R-1} \tag{6}$$

$$M(\theta) \simeq \frac{p^2\, e^{\theta(R+2)}}{(1 - e^{\theta} q)^2} \tag{7}$$

$$\mu_N \simeq (R + 2) + \frac{2q}{p} \tag{8}$$

$$\sigma_N^2 \simeq \frac{2q}{p^2} \tag{9}$$

We note also in passing that for large $R (\geq 18)$ the mean number of days to launch the three vehicles (μ_N) is approximately equal to the median number of days. (The median number of days is such that the probability is 0.5 that the three vehicles will be launched on or before that day.) This number is given as the abscissa of the curve for which $F(N) = 0.5$ (Fig. 1 and 2). The approximation is not too poor even for small R. Thus, for example, we obtain Table 1 from Eq. (8) and Fig. 1 and 2.

Table 1. A comparison of the mean and median number of days

R	p	0.2	0.3	0.4
18	μ_N	28	25^-	23
	Median	26	25^+	24^-
1	μ_N	15	10	7.5
	Median	13	9	6

V. DERIVATION OF PRINCIPAL RESULTS

Under the assumption of Section II, the probability of the first successful countdown on the kth trial for any vehicle is clearly

$$p_k = pq^{k-1} \qquad k = 1,2,\cdots \tag{10}$$

(i.e., the Pascal or geometric distribution). Suppose now that we label the pads 1 and 2 and the first vehicle on each pad by the same number, the standby vehicle being labeled 3. Then, there is only one possible way in which the vehicles may be launched. Specifically, the order of launching is 1-2-3. However, there are two distinct and mutually exclusive cases to consider:

Case	*Event*
1	No. 2 goes before turnaround time is completed
2	No. 2 goes at, or after completion of, turnaround time

Now, the probability of three successful countdowns in N days, given case 1, is

$$P\{N \text{ days} \mid \text{case } 1\} = \sum_{k=1}^{N-R-1} \sum_{m=1}^{R-1} P \left\{ \begin{array}{l} \text{No. 1 took } k \text{ trials,} \\ \text{No. 2 took } m \text{ trials, and} \\ \text{No. 3 took } N-k-R \\ \text{trials} \end{array} \right\} \tag{11}$$

where the limits for m and k are easily understood from the following sketch:

$\longleftarrow$ N days $\longrightarrow$

k days	R days	$N-R-k$ days
No. 1	No. 2	No. 3

$1 \leq m \leq R-1$ trials at most available to No. 2

$\longleftarrow$ N days $\longrightarrow$

k days	R days	1 day
No. 1	No. 2	No. 3

$1 \leq k \leq N-R-1$ trials at most available to No. 1

From the assumed independence of the vehicles, the joint event appearing in the summation of Eq. (11) can be written in the form

$$\begin{aligned} P\{N \text{ days} \mid \text{case } 1\} &= \sum_{k=1}^{N-R-1} \sum_{m=1}^{R-1} pq^{k-1}\, pq^{m-1}\, pq^{N-k-R-1} \\ &= \sum_{k=1}^{N-R-1} p^2 q^{N-R-2} \sum_{m=1}^{R-1} pq^{m-1} \\ &= \sum_{k=1}^{N-R-1} p^2 q^{N-R-2} (1 - q^{R-1}) \\ &= p^2 q^{N-R-2} (1 - q^{R-1})(N - R - 1) \end{aligned} \tag{12}$$

Similarly, for case 2, we have

$$P\{N \text{ days} \mid \text{case } 2\} = \sum_{k=1}^{N-R-1} \sum_{m=R}^{N-k-1} P \begin{Bmatrix} \text{No. 1 took } k \text{ trials,} \\ \text{No. 2 took } m \text{ trials, and} \\ \text{No. 3 took } N-k-m \text{ trials} \end{Bmatrix} \tag{13}$$

where, as before, the limits for m and k are easily deduced from a consideration of the following sketch:

$\longleftarrow$ N days $\longrightarrow$

k days	R days	$m-R$ days	$N-k-m$ days
No. 1		No. 2	No. 3

$R \leq m \leq N-k-1$ trials
at most available to No. 2

$\longleftarrow$ N days $\longrightarrow$

k days	$R-1$ days	1 day	1 day
No. 1		No. 2	No. 3

$1 \leq k \leq N-R-1$ trials
at most available to No. 1

Thus,

$$\begin{aligned} P\{N \text{ days} \mid \text{case } 2\} &= \sum_{k=1}^{N-R-1} \sum_{m=R}^{N-k-1} p_k \, p_m \, p_{N-k-m} \\ &= \sum_{k=1}^{N-R-1} pq^{k-1} \sum_{m=R}^{N-k-1} pq^{m-1} \, pq^{N-k-m-1} \\ &= \sum_{k=1}^{N-R-1} p^3 q^{N-3} (N-k-R) \qquad (14) \\ &= p^3 q^{N-3} \left[(N-R)(N-R-1) - \frac{(N-R)(N-R-1)}{2} \right] \\ &= \frac{p^3 q^{N-3}}{2} \, (N-R)(N-R-1) \end{aligned}$$

The sum of Eq. (12) and (14) is the answer $f(N)$ we are seeking.

The derivations of the cumulative distribution function, the moment-generating function, and the means and variance are straightforward from the definitions once $f(N)$ is known and involve only summing geometric and related series. The details are left to the interested reader.

The approximate formulas of Section IV are obtained by deleting all terms with a factor of q^R or q^N (but not terms like q^{N-R}). Physically, this is equivalent to the rather obvious fact that for large R, case 2 of this Section is a second-order effect. That is, the probability of failing to launch vehicle 2 in the turnaround time is very small.